

## Sage Lecture 1. Introduction to Sage

Sage is an open source software system that bundles many standard open source mathematical software libraries into a common Python-based interface. Cloud computing is well supported: you do not have to install Sage on your computer to compute.

### 1.1 The Sage environment

We will use the online version of Sage, so the GUI is your web browser.

Three steps to getting started:

1. Go to <http://www.sagemath.org>
2. click on **Try it online!**
3. Sign up for a new SAGE notebook account.

In this lecture we examine the following problem: how do two circles intersect?

### 1.2 An Intersection Problem

We start out plotting and solving specific instances of the two circle problem: Consider two circles, how do they intersect?

```
sage: c1 = circle( (1,2) , 3 , rgbcolor=(1,0,0) )
sage: c2 = circle( (-1,1) , 2 , rgbcolor=(0,0,1) )
sage: c12 = c1 + c2
sage: c12.show(aspect_ratio=1)
```

To compute the intersection points, we formulate the algebraic problem: solve a polynomial system. We solve a system of two polynomial equations in two unknowns:

```
sage: x,y = var('x, y')
sage: p1 = (x-1)^2 + (y-2)^2 - 3^2 == 0
sage: p2 = (x+1)^2 + (y-1)^2 - 2^2 == 0
sage: sols = solve([p1,p2],x,y)
sage: sols
```

We obtain two solutions in symbolic form. Let us verify the solutions:

```
sage: print sols[1]
sage: vx = sols[1][0].rhs()
sage: print n(vx,200)
sage: vy = sols[1][1].rhs()
sage: s = p1.substitute(x=vx,y=vy)
sage: print s
sage: s.expand()
```

Likewise we do it for the second solution and also for  $p2$ .

Let us consider the general problem formulation and examine the choice of the coordinate system. Without loss of generality we may choose (1) the origin is center of the first circle; and (2) the radius of the first circle to be one. Thus the first circle is the unit circle:  $f = x^2 + y^2 - 1$ .

We may choose the orientation of the  $x$ -axes

- (1) through the center of the second circle:  $(c, 0)$ ;
- (2) let  $r$  be the radius of the second circle.

Thus we have two parameters for the second circle:  $g = (x - c)^2 + y^2 - r^2$ .

Our problem is governed by two parameters:  $c$  and  $r$ .

To obtain a general solution we apply symbolic computation (manipulate symbols as numbers) and declare  $c$  and  $r$  as variables and solve:

```
sage: c,r = var('c, r')
sage: f = x^2 + y^2 - 1
sage: g = (x-c)^2 + y^2 - r^2
sage: solve([f==0,g==0],x,y)
```

We obtain a symbolic solution. But how general is this solution?

### 1.3 Determinants, Resultants and Discriminants

Possible singular solutions are

- (1) double solutions: two circles touching each other; or
- (2) a solution set: two overlapping circles.

At a singular solution the determinant of the Jacobian matrix vanishes.

The Jacobian matrix collects all partial derivatives:

```
sage: J = matrix([[diff(f,x),diff(f,y)],\
                  [diff(g,x),diff(g,y)]])
sage: print J
sage: dJ = det(J)
sage: print dJ
```

The determinant provides a condition for a linear system to have a singular solution. In particular: the determinant of the coefficient matrix  $A$  is zero if and only if the system  $A\mathbf{x} = \mathbf{b}$  has no unique solution.

The Discriminant generalizes the determinant to polynomial systems. We will compute the discriminant of the circle problem. The discriminant is a polynomial in  $c$  and  $r$  which will vanish whenever the solutions to the circle problem are singular.

Adding the determinant of the Jacobian matrix to the system, we solve

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ (x - c)^2 + y^2 - r^2 = 0 \\ 4cy = 0 \end{cases}$$

Looking long enough at the system will lead to the solutions... Our aim is to illustrate a general approach.

The resultant is a tool to eliminate variables and to reduce the solution of polynomial systems in several variables to problems in one variable. To introduce the resultant consider the following two polynomials two polynomials  $p$  and  $q$  with general coefficients:

$$\begin{aligned} p(x) &= a_2x^2 + a_1x + a_0 \\ q(x) &= b_2x^2 + b_1x + b_0 \end{aligned}$$

For which values of the coefficients do  $p$  and  $q$  have a common factor?

The resultant is a polynomial in the coefficients of  $p$  and  $q$  which is zero for those coefficients for which  $p$  and  $q$  have a common factor. Our application of the resultant is to eliminate  $x$ .

Suppose  $p$  and  $q$  do have a factor  $f$ :  $p = Pf$ ,  $q = Qf$ . Observe:  $Qp = QPf$  and  $Pq = PQf$  imply  $Qp = Pq$ .

$$\begin{aligned} p(x) &= a_2x^2 + a_1x + a_0 & P(x) &= \alpha_1x + \alpha_0 \\ q(x) &= b_2x^2 + b_1x + b_0 & Q(x) &= \beta_1x + \beta_0 \end{aligned}$$

Elaborate the condition  $Qp = Pq$  and consider

$$(\beta_1 + \beta_0)(a_2x^2 + a_1x + a_0) = (\alpha_1x + \alpha_0)(b_2x^2 + b_1x + b_0)$$

$$\begin{array}{rclcl} x^3: & \beta_1 a_2 & & = & \alpha_1 b_2 \\ x^2: & \beta_1 a_1 + & \beta_0 a_2 & = & \alpha_1 b_1 + \alpha_0 b_2 \\ x^1: & \beta_1 a_0 + & \beta_0 a_1 & = & \alpha_1 b_0 + \alpha_0 b_1 \\ x^0: & & \beta_0 a_0 & = & \alpha_0 b_0 \end{array}$$

Resultants are determinants. We solve a linear system in  $\beta_1$ ,  $\beta_0$ ,  $\alpha_1$ , and  $\alpha_0$ :

$$\begin{cases} \beta_1 a_2 & = & \alpha_1 b_2 \\ \beta_1 a_1 + \beta_0 a_2 & = & \alpha_1 b_1 + \alpha_0 b_2 \\ \beta_1 a_0 + \beta_0 a_1 & = & \alpha_1 b_0 + \alpha_0 b_1 \\ \beta_0 a_0 & = & \alpha_0 b_0 \end{cases}$$

In matrix form:

$$\begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_0 \\ -\alpha_1 \\ -\alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Condition for nonzero solution: determinant of matrix is zero. We can use this condition as a computational tool. For example, consider the solution of the quadratic equation:  $ax^2 + bx + c = 0$ .

```
sage: R.<x,a,b,c> = QQ[]
sage: p = a*x^2 + b*x + c
sage: dp = diff(p,x)
sage: disc = singular.resultant(p,dp,x)
sage: print disc
sage: print factor(R(disc))
```

The discriminant is a polynomial in the coefficients and vanishes whenever the polynomial and its derivative have a common solution.

To compute resultants, we first declare a polynomial ring with rational coefficients:

```
sage: R.<x,y,c,r> = QQ[]
sage: F = R(f)
sage: G = R(g)
sage: D = R(dJ)
```

We can now eliminate  $x$  using the resultant from **Singular**.

```
sage: rFG = singular.resultant(F,G,x)
sage: print rFG
sage: rGD = singular.resultant(G,D,x)
sage: print rGD
```

The discriminant of our circle problem gives an algebraic condition on the parameters of the problem

```
sage: discriminant = singular.resultant(rFG,rGD,y)
sage: print discriminant
```

The geometric interpretation is as follows. The discriminant gives the relation between center  $(c,0)$  and radius  $r$  of the second circle for which the solutions are singular, i.e.: (1) double solutions: circles touch each other; and (2) a solution set: overlapping circles.

To simplify the condition on the parameters we factor the discriminant. To factor the discriminant, we must convert to an element of the ring  $R$ .

```
sage: print type(discriminant)
sage: factor(R(discriminant))
```

shows

```
<class 'sage.interfaces.singular.SingularElement'>
(256) * (c - r - 1)^2 * (c - r + 1)^2
      * (c + r - 1)^2 * (c + r + 1)^2 * c^4
```

So the discriminant for our problem looks as follows:

$$256(c-r-1)^2(c-r+1)^2(c+r-1)^2(c+r+1)^2c^4$$

Collecting all formulas, the system

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ (x-c)^2 + y^2 - r^2 = 0 \end{cases}$$

has exactly two solutions

$$\left[ x = \frac{-r^2 + c^2 + 1}{2c}, y = \pm \frac{\sqrt{-r^4 + 2c^2r^2 + 2r^2 - c^4 + 2c^2 - 1}}{2c} \right]$$

except for those  $c$  and  $r$  satisfying

$$256(c-r-1)^2(c-r+1)^2(c+r-1)^2(c+r+1)^2c^4 = 0.$$

The discriminant variety is the solution set of  $256(c-r-1)^2(c-r+1)^2(c+r-1)^2(c+r+1)^2c^4 = 0$ . Considering only the positive values for  $c$  and  $r$ , we classify the regular solutions in four different configurations.

## 1.4 Assignments

1. Go to [www.sagemath.org](http://www.sagemath.org) and take the Feature Tour. What topics do interest you most? Make a Sage notebook and execute what you learned on the tour.
2. Consider the system  $\begin{cases} x^2 + y^2 - 5 = 0 \\ 9x^2 + y^2 - 9 = 0. \end{cases}$  Give all Sage commands to solve (a) and (b) below:
  - (a) Make a plot of the two algebraic curves.
  - (b) Solve the system. How many solutions do you find?
3. Consider the system  $\begin{cases} x^2 + y^2 - 5 = 0 \\ 9x^2 + \alpha y^2 - 9 = 0. \end{cases}$  Use Sage to find the value for the parameter  $\alpha$  for which the curves not intersect but *touch* each other. Write all relevant Sage commands below and give the critical value for  $\alpha$  as well.